Mass energy relation

Study Material By: Sunil Kumar Yadav*

Department of Physics,

Maharaja College, Ara,

Bihar 802301, India.

(Dated: May 8, 2020)

"Matter is Energy ... Energy is Light ... We are all Light Beings"

— Albert Einstein

In the previous article, we have discussed variation of mass with velocity and how classical definition of momentum changed in Einstein's special relativity theory. In the present article, we study the mass energy relation and energy momentum relation in special relativity theory.

Let us start with Newtonian definition of kinetic energy (K)

$$K = \int_{u=0}^{u=u} \mathbf{F}.d\mathbf{x}, \tag{1}$$

which describe the work done by external force in increasing the particle speed from zero to u. For one dimensional case the above equation can be written as

$$\begin{aligned} K &= \int_{u=0}^{u=u} F dx \\ &= m_0 \int_{u=0}^{u=u} \frac{du}{dt} dx \\ &= m_0 \int_{u=0}^{u=u} \frac{dx}{dt} du \\ &= m_0 \int_{u=0}^{u=u} u du \\ &= \frac{1}{2} m_0 u^2, \end{aligned} \tag{2}$$

where m_0 is the Newtonian mass or the rest mass. We have got the

*Electronic address: sunil.phy30@gmail.com

Next, we take mass to be relativistic mass (which we have discussed in the previous article) and apply the same definition for the kinetic energy

$$
\begin{aligned}
K &= \int_{u=0}^{u=u} F dx \\
&= \int_{u=0}^{u=u} \frac{d(mu)}{dt} dx \\
&= \int_{u=0}^{u=u} \frac{dx}{dt} d(mu) \\
&= \int_{u=0}^{u=u} u(mdu + udm) \\
&= \int_{u=0}^{u=u} (mudu + u^2 dm), \qquad (3)
\end{aligned}
$$

where m is the relativistic mass defined by $m = \gamma m_0$ with $\gamma = 1/\sqrt{1 - u^2/c^2}$. Here it is to be noted that m and u are the variables. Next, we need to rewrite integrand in terms of one variable. Therefore, we rewrite the relativistic mass definition as

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \qquad (4)$$

or

$$m^2c^2 - m^2u^2 = m_0^2c^2. \qquad (5)$$

Now taking differential of the above equation, we obtain

$$2mc^2dm - 2mu^2dm - 2m^2udu = 0$$

or

$$mudu + u^2dm = c^2dm \qquad (6)$$

Now we use above result in Eq. (3)

$$K = \int_{u=0}^{u=u} c^2 dm = \int_{m=m_0}^{m=m} c^2 dm = mc^2 - m_0c^2, \qquad (7)$$

where we have used Eq. (4) for changing the limit. We can rewrite Eq. (7) as

$$\boxed{K = \gamma m_0 c^2 - m_0 c^2 = (\gamma - 1) m_0 c^2}, \qquad (8)$$

which is the relativistic expression for kinetic energy. Now if we define $E = \gamma m_0 c^2$, where E is the total energy of the object, we can write

$$\boxed{E = m_0c^2 + K = E_0 + K}, \qquad (9)$$

where $E_0 = m_0c^2$ is the rest energy of the particle.

Energy Momentum Relation: Next, we establish a relation between Total energy E and momentum p of the particle. The relativistic momentum is given by

$$p = \frac{m_0 u}{\sqrt{1 - u^2/c^2}} \tag{10}$$

and total energy by

$$E = \frac{m_0 c_2}{\sqrt{1 - u^2/c^2}}. \tag{11}$$

By eliminating u from Eqs. (10) and (11) we can obtain required relation between E and p. Next, squaring Eqs. (10) and (11), we obtain

$$p^2c^2 = \frac{m_0^2 u^2 c^2}{1 - u^2/c^2} \tag{12}$$

$$E^2 = \frac{m_0^2 c^4}{1 - u^2/c^2}. \tag{13}$$

Now subtracting Eq. (12) from Eq. (13), we obtain

$$E^2 - (pc)^2 = (m_0c^2)^2 \tag{14}$$

or

$$\boxed{E^2 = (pc)^2 + (m_0c^2)^2}. \tag{15}$$

The important point to note here that, in Eq. (14), m_0c^2 is invariant, therefore, the difference $E^2 - (pc)^2$ is also invariant.

---

[1] R. Resnick, *Introduction to Special Relativity*, Wiley-VCH, (1968) .

[2] A. Beiser, *Concepts of modern physics*, Tata McGraw-Hill Education, (2003).